Journal of Pure and Applied Algebra 35 (1985) 123-132
North-Holland

SUBGROUPS OF THE ELEMENTARY AND STEINBERG GROUPS OF CONGRUENCE LEVEL I^2

Susan C. GELLER*
Mathematics Department, Texas A & M University, College Station, TX 77843, USA

Charles A. WEIBEL**
Mathematics Department, Rutgers University, New Brunswick, NJ 08903, USA

Communicated by H. Bass
Received 21 August 1983

There are several subgroups of the Elementary group $E(B,I)$ and of the Steinberg group $\mathrm{St}(B,I)$ which lie at the congruence level I^2. We describe their interrelationships and give examples to show that they can be distinct. Our main result is that, although the group $E(B,I)$ depends on the choice of the ambient ring B, its commutator subgroup does not. In fact, the commutator subgroup is the intersection of $\mathrm{Gl}(I^2)$ with $E(Z\oplus I, I)$.

Attached to an ideal I of a ring B are two important subgroups of the infinite general linear group $\mathrm{Gl}(B)$: the group $\mathrm{Gl}(I)=\mathrm{Gl}(B,I)$ of invertible matrices congruent to 1 mod I, and the relative elementary group $E(B,I)$ (the normal subgroup generated by the matrices $e_{ij}(x)$, x in I).

Whereas $\mathrm{Gl}(I)$ may be defined intrinsically to I, it is well-known that the subgroup $E(B,I)$ of $\mathrm{Gl}(I)$ varies with the choice of the ambient ring B. The 'smaller' B is, the smaller $E(B,I)$ is. The 'smallest' choice of B, namely $B=\mathbb{Z}\oplus I$, gives rise to the smallest such subgroup, namely $E(I)=E(\mathbb{Z}\oplus I,I)$. The main result of this paper is that the dependence on B vanishes when we pass to commutator subgroups:

Theorem 1. *The following subgroups of* $\mathrm{Gl}(I)$ *are equal:*

$$[E(I),E(I)]=[E(B,I),E(B,I)]=[\mathrm{Gl}(B,I),\mathrm{Gl}(B,I)].$$

AMS-MOS (1980) Subj. Class.: Primary 18F25; Secondary 13D15, 16A54.
Keywords and Phrases: Elementary group, relative Steinberg groups, congruence level subgroup, algebraic K-theory.

* Partially supported by NSF grant MCS82-00564.
** Partially supported by NSF grant MCS81-02753..

We also have $[E(I), E(I)] = E(I) \cap \mathrm{Gl}(I^2)$. *However, in general*

$$E(I) \cap \mathrm{Gl}(I^2) \neq E(B, I) \cap \mathrm{Gl}(I^2).$$

To fit this result into another framework, recall that a subgroup H of $\mathrm{Gl}(B)$ is said to be of *congruence level* I^2 if $E(B, I^2) \subseteq H \subseteq \mathrm{Gl}(B, I^2)$. It is classical that $[E(B, I), E(B, I)]$ is of congruence level I^2 (see [1, p. 223] for example). Theorem 1 may be interpreted as a statement about subgroups of congruence level I^2. Another subgroup of congruence level I^2 is the group $E(I^\infty, 1, I^\infty)$. This is the subgroup $\mathrm{Gl}(I)$ generated by matrices $1 + vw$, where v is a column vector with entries in I, w is a row vector with entries in I, and $wv = 0$ under the usual inner product. Our second result is this:

Theorem 2. *There is a chain of inclusions of subgroups of congruence level* I^2:

$$E(B, I^2) \subseteq E(I^\infty, 1, I^\infty) \subseteq [E(I), E(I)] = E(I) \cap \mathrm{Gl}(I^2)$$
$$\subseteq E(B, I) \cap \mathrm{Gl}(I^2) \subseteq \mathrm{Gl}(I^2).$$

All groups involved are normal subgroups of $\mathrm{Gl}(B)$, *and all inclusions can be strict.*

We can lift the above notions to the relative Steinberg groups $\mathrm{St}(B, I)$ and obtain analogous but more delicate results. (See [5], [6] for the definition of $\mathrm{St}(B, I)$.) These results are more complicated to state because there is no ambient group such as $\mathrm{Gl}(I)$, and because $\mathrm{St}(B, I^2)$ is not even a subgroup of $\mathrm{St}(B, I)$. To circumvent these difficulties, we write $\mathrm{St}(B, I) \cap \mathrm{Gl}(I^2)$ for the subgroup of $\mathrm{St}(B, I)$ mapping into $\mathrm{Gl}(I^2)$ under $\mathrm{St}(B, I) \to \mathrm{Gl}(I)$, and we will utilize a $\overline{\text{bar}}$ over groups to indicate that we mean their images in $\mathrm{St}(B, I)$. We abbreviate $\mathrm{St}(\mathbb{Z} \oplus I, I)$ as $\mathrm{St}(I)$. The subgroup $[\mathrm{Gl}(B, I), \mathrm{St}(B, I)]$ of the $\mathrm{Gl}(B)$-group $\mathrm{St}(B, I)$ is generated by the elements $(g \cdot X)X^{-1}$ with $g \in \mathrm{Gl}(B, I)$ and $X \in \mathrm{St}(B, I)$. (The action of $\mathrm{Gl}(B)$ is described in 8.2 below, as well as in [4, (3.8)].) A subgroup H of $\mathrm{St}(B, I)$ is said to be of *congruence level* I^2 if

$$\overline{\mathrm{St}(B, I^2)} \subseteq H \subseteq \mathrm{St}(B, I) \cap \mathrm{Gl}(I^2).$$

To indicate the delicacy of the problem, consider the special case $I^2 = 0$. Since $\mathrm{St}(B, I) \cap \mathrm{Gl}(I^2) = K_2(B, I)$, a subgroup of $\mathrm{St}(B, I)$ is of congruence level I^2 exactly when it is contained in $K_2(B, I)$. The situation is described by the following result:

Theorem 3. *When* $I^2 = 0$, *the nilpotent group* $\mathrm{St}(B, I)$ *need not be abelian. The commutator subgroup is*

$$\overline{\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)} = \overline{\mathrm{St}(I^\infty, B, I^\infty)} = [\overline{\mathrm{St}(I)}, \overline{\mathrm{St}(I)}]$$
$$= [\mathrm{St}(B, I), \mathrm{St}(B, I)] = [\mathrm{Gl}(B, I), \mathrm{St}(B, I)]$$
$$= \overline{K_2(\mathbb{Z} \oplus I, I)}.$$

Furthermore, all inclusions can be strict in the chain

$$1 \subsetneq [\mathrm{St}(B,I),\mathrm{St}(B,I)] \subsetneq K_2(B,I).$$

When $I^2 \neq 0$, the situation is even more interesting. To begin with, $\overline{\mathrm{St}(B,I^2)}$ is no longer trivial, and $\mathrm{St}(B,I) \cap \mathrm{Gl}(I^2)$ is strictly bigger than $K_2(B,I)$. In addition, the equalities of Theorem 3 break down somewhat.

Theorem 4. *The following relations hold in* $\mathrm{St}(B,I)$:

(i) $\overline{\mathrm{St}(B,I^2)} = \ker(\mathrm{St}((B,I) \to \mathrm{St}(B/I^2, I/I^2))$.

(ii) $\overline{\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)} = \overline{\mathrm{St}(I^\infty, B, I^\infty)}$. (*These two groups are defined in paragraph* 8 below.)

(iii) $[\overline{\mathrm{St}(I)}, \overline{\mathrm{St}(I)}] = [\mathrm{St}(B,I),\mathrm{St}(B,I)] = [\mathrm{Gl}(B,I),\mathrm{St}(B,I)] = \overline{\mathrm{St}(I)} \cap \mathrm{Gl}(I^2)$.

(iv) $\overline{K_2(\mathbb{Z} \oplus I, I)}$ *lies in the commutator subgroup of* $\mathrm{St}(B,I)$.

There is also the following chain of inclusions:

$$\overline{\mathrm{St}(B,I^2)} \subsetneq \overline{\mathrm{St}(I^\infty, B, I^\infty)} \subsetneq [\mathrm{St}(B,I),\mathrm{St}(B,I)] \subsetneq \mathrm{St}(B,I) \cap \mathrm{Gl}(I^2).$$

All groups involved are normal subgroups of $\mathrm{St}(B,I)$, *and all inclusions can be strict.*

Having stated our results, we now turn to the task of proving them. We first prove Theorems 1 and 2.

One standard approach to studying subgroups of congruence level I^2 is to compute inside the quotient group $K_1(B,I^2) = \mathrm{Gl}(B,I^2)/E(B,I^2)$. We illustrate this technique with an example.

Example 5. To see that $E(I) \cap \mathrm{Gl}(I^2) \neq E(B,I) \cap \mathrm{Gl}(I^2)$, consider $B = k[t]$ and $I = tB$, where k is a field with $\Omega_k \neq 0$. Set $A = \mathbb{Z} \oplus I$. Since $K_*(B,I) = 0$, we see that $E(B,I) = \mathrm{Gl}(I)$ and that $E(B,I)/E(A,I) = K_1(A,I)$. From [4], we see that $K_1(A,I) = K_1(A,B,I) = \Omega_k \neq 0$. On the other hand, since I/I^2 is nilpotent we have $SK_1(A/I^2, I/I^2) = 0$. Thus $E(A/I^2, I/I^2) = E(B/I^2, I/I^2) = \mathrm{Sl}(I/I^2)$. From the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E(A,I) \cap \mathrm{Gl}(I^2) & \longrightarrow & E(A,I) & \longrightarrow & E(A/I^2, I/I^2) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & E(B,I) \cap \mathrm{Gl}(I^2) & \longrightarrow & E(B,I) & \longrightarrow & E(B/I^2, I/I^2) & \longrightarrow & 1 \end{array}$$

we see that the cokernel of $E(A,I) \cap \mathrm{Gl}(I^2) \to E(B,I) \cap \mathrm{Gl}(I^2)$ is isomorphic to $\Omega_k \neq 0$. Note that as a byproduct of this computation (and Theorem 1) we obtain a direct sum decomposition of the abelianization of $\mathrm{Gl}(I)$:

$$\mathrm{Gl}(I)/[\mathrm{Gl}(I),\mathrm{Gl}(I)] = \mathrm{Sl}(I/I^2) \oplus \Omega_k.$$

5.1. If we replace $A=\mathbb{Z}\oplus I$ by $A=k_0\oplus I$, where k_0 is a subfield of k, the above argument goes through with the sole change that the cokernel of $E(A,I)\cap \mathrm{Gl}(I^2)\to E(B,I)\cap \mathrm{Gl}(I^2)$ is isomorphic to $K_1(A,I)\cong\Omega_{k/k_0}$. This shows that in general we have

$$E(A,I)\cap \mathrm{Gl}(I^2)\neq \mathrm{Gl}(A,I^2).$$

Proof of Theorem 1. Let X, Y be elements of $\mathrm{Gl}_n(B,I)$. In $\mathrm{Gl}_{3n}(B,I)$ we have

$$[X,Y]=\left[\begin{pmatrix} X & 0 & 0\\ 0 & X^{-1} & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} Y & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & Y^{-1}\end{pmatrix}\right]$$

and both of the right-hand matrices are in $E_{3n}(I)$. (See [1, p. 227].) This shows that $[E(I),E(I)]=[\mathrm{Gl}(B,I),\mathrm{Gl}(B,I)]$. Example 5 above shows that the evident inclusion $E(I)\cap \mathrm{Gl}(I^2)\subseteq E(B,I)\cap \mathrm{Gl}(I^2)$ can be strict. It remains to show that the inclusion of $[E(I),E(I)]$ in $E(I)\cap \mathrm{Gl}(I^2)$ is an equality. We can consider this problem inside $K_1(B,I^2)$, where $B=\mathbb{Z}\oplus I$. The group $E(I)\cap \mathrm{Gl}(I^2)/E(B,I^2)$ is the kernel of $K_1(B,I^2)\to K_1(B,I)$, i.e., the image of the boundary map $\partial: K_2(B/I^2, I/I^2)\to K_1(B,I^2)$ in the long exact sequence associated to $I^2\subset I$. By [2, (6.5)], the group $K_2(B/I^2,I/I^2)$ is generated by the following elements of $\mathrm{St}(B/I^2,I/I^2)$:

$$\langle\bar{x},\bar{y}\rangle=H_{12}(\bar{x},\bar{y})h_{12}(1+\overline{xy})^{-1}\qquad (\bar{x},\bar{y}\text{ in }I/I^2)$$
$$=H_{12}(\bar{x},\bar{y})=[x_{21}(-\bar{y}),x_{12}(\bar{x})].$$

(See [2, p. 126] for the notation used here.) To compute $\partial\langle\bar{x},\bar{y}\rangle$, we lift $\langle\bar{x},\bar{y}\rangle$ to the element $[x_{21}(-y),x_{12}(x)]$ of $\mathrm{St}(B,I)$ and consider its image in $\mathrm{Gl}(I^2)$ under $\mathrm{St}(B,I)\to \mathrm{Gl}(I)$. The element $\partial\langle\bar{x},\bar{y}\rangle$ in $K_1(B,I^2)$ is the class of the matrix

$$[e_{21}(-y),e_{12}(x)]=\begin{pmatrix} 1+xy & -xyx\\ -yxy & 1-yx+yxyx\end{pmatrix}.$$

Therefore $E(I)\cap \mathrm{Gl}(I^2)$ is generated by $E(I^2)$, together with the commutators $[e_{21}(-y),e_{12}(x)]$, i.e., $E(I)\cap \mathrm{Gl}(I^2)=[E(I),E(I)]$.

Proof of Theorem 2. To avoid repetition, we will deduce the chain of inclusions in 11.3 from the corresponding inclusions in the Steinberg group. It is well-known [1, (V.2.1.b), p. 229] that $\mathrm{Gl}(I^2)$, $E(B,I)$ and $E(B,I^2)$ are normal subgroups of $\mathrm{Gl}(B)$. The subgroup $E(I^\infty,1,I^\infty)$ is also normal in $\mathrm{Gl}(B)$ because its set of generators is closed under conjugation. This implies normality of all groups listed. Examples 6 and 7 below show that the first two inclusions can be strict. We have already established strictness of the last two inclusions in 5 and 5.1.

6. To see that $E(B,I^2)\neq E(I^\infty,1,I^\infty)$ as subgroups of $\mathrm{Gl}(I^2)$, it is enough to see that the image of $E(I^\infty,1,I^\infty)$ in $\mathrm{Gl}(I^2)/E(B,I^2)=K_1(B,I^2)$ can be nonzero. We give two examples of this phenomenon.

Example 6.1. When B is commutative and $x, y \in I$ consider the element

$$1+\begin{pmatrix} x \\ y \end{pmatrix}(-y\ \ x)=\begin{pmatrix} 1-xy & x^2 \\ -y^2 & 1+xy \end{pmatrix}$$

of $E(I^\infty, 1, I^\infty)$. Its image in $K_1(B, I^2)$ is the Mennicke symbol

$$\begin{bmatrix} x^2 \\ 1-xy \end{bmatrix}$$

which is well-known to be nonzero in general.

For example, let $B=\mathbb{Q}[x,y]$, $I=(x,y)$. Then by [10, (1.4)] we have isomorphisms

$$\mathbb{Q} \cong \Lambda^2(I/I^2) \cong K_2(B/I^2, I/I^2) \overset{\partial}{\cong} K_1(B, I^2)$$

and $2 \in \mathbb{Q}$ corresponds to the Mennicke symbol

$$\partial(\langle x, y\rangle)^2=\begin{bmatrix} x^2 y \\ 1-xy \end{bmatrix}^2=\begin{bmatrix} x^2 \\ 1-xy \end{bmatrix}.$$

Example 6.2. Here is a noncommutative example with $I^3=0$. Let $B=\mathbb{Z}\{x,y\}/(x^2, y^2, yx)$ and $I=(x,y)B$. Then by [7, (2.6)] the Dieudonné determinant is an isomorphism:

$$\begin{array}{ccccc} K_1(B, I^2) & \cong & I^2/[B, I^2] & \cong & \mathbb{Z}. \\ 1+axy & \leftrightarrow & axy & \leftrightarrow & a \end{array}$$

This shows that $E(B, I^2) \neq E(I^\infty, 1, I^\infty)$, because if $a \neq 0$ the element $1+(ax)y$ is in $E(I^\infty, 1, I^\infty)$ but not in $E(B, I^2)$.

Example 7. When $B=\mathbb{Z}[t]$, $I=tB$, we claim that

$$E(B, I^2)=E(I^\infty, 1, I^\infty) \neq [E(I), E(I)].$$

To see this, note that the class of the commutator

$$[e_{21}(1;t), e_{12}(t)]=\begin{pmatrix} 1+t^2-t^3 & * \\ -t^3 & * \end{pmatrix}$$

in $\mathrm{Gl}(I^2)/E(B, I^2)=K_1(B, I^2)$ is the Mennicke symbol

$$\begin{bmatrix} -t^3 \\ 1+t^2 \end{bmatrix}^{-1}=\begin{bmatrix} -t^3 \\ 1-t^2 \end{bmatrix}.$$

By [3, p. 271], $K_2(\mathbb{Z}[t]/(t^2), t)$ is nonzero and is generated by $\langle t, t\rangle$. From the ideal sequence it follows that $K_1(B, I^2) \cong K_2(\mathbb{Z}[t]/(t^2), t)$ and that

$$\partial\langle t, t\rangle=\begin{bmatrix} -t^3 \\ 1-t^2 \end{bmatrix}.$$

This shows that $E(B, I^2) \neq [E(I), E(I)]$.

On the other hand, $E(I^\infty, B, I^\infty)$ is generated by symbols $1+(vt)(tw)=1+t^2vw$ with $(tw)(vt)=0$, and hence with $wv=0$ (t is a nonzerodivisor). By [8], the element $1+t^2vw$ belongs to $E(B, I^2)$. Hence $E(I^\infty, B, I^\infty)=E(B, I^2)$ as claimed.

We now turn to the proofs of Theorems 3 and 4. To do so will require some facility with the groups $\mathrm{St}(B, I)$. In addition, we need to define the groups $\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)$ and $\mathrm{St}(I^\infty, B, I^\infty)$. These are special cases of the more general family of groups $\mathrm{St}(V, J, W)$ which van der Kallen defined in [9]. As we will use other groups in this family, we need to recall the definition of the groups $\mathrm{St}(V, J, W)$.

8. Let B be an associative ring with unit, and J a 2-sided ideal of B. Let V be a right B-module, W a left B-module. Every bilinear form $W \otimes V \to B$ gives rise to a group $\mathrm{St}(V, J, W)$, defined by the following presentation.

Generators: $X(v, x, w)$, where $(v, x, w) \in V \times J \times W$ and $w \cdot v = 0$.

Relations:
(a) $X(v, bx, w) = X(vb, x, w), \quad b \in B,$
(b) $X(v, xb, w) = X(v, x, bw), \quad b \in B,$
(c) $X(v, x_1 + x_2, w) = X(v, x_1, w)X(v, x_2, w),$
(d) $X(v_1 + v_2, x, w) = X(v_1, x, w)X(v_2, x, w)$ if $w \cdot v_i = 0,$
(e) $X(v, x, w_1 + w_2) = X(v, x, w_1)X(v, x, w_2)$ if $w_i \cdot v = 0,$
(f) $X(v', x', w')X(v, x, w)X(v', x', w')^{-1}$
$= X(v + v'x'(w' \cdot v), x, w - (w \cdot v')x'w').$

Scholium 8.1. $\mathrm{St}(V, J, W)$ acts by transvections on the left of V: $X(v', x, w') \cdot v = v + v'x(w' \cdot v)$. Similarly, $\mathrm{St}(V, J, W)$ acts by transvections on the right of W. Using these actions, we may rewrite the relations as:
(a–e) The symbols $X(v, x, w)$ are trilinear.
(f) $gX(v, x, w)g^{-1} = X(gv, x, wg^{-1})$.

Example 8.2. Let B^∞ denote the free B-module which has basis $\{e_1, e_2, \ldots\}$. The bilinear form $B^\infty \otimes B^\infty \to B$ is given by the matrix product wv, thinking of w as a row vector, and of v as a column vector. The resulting group $\mathrm{St}(B^\infty, B, B^\infty)$ is isomorphic to $\mathrm{St}(B)$, with $x_{ij}(b)$ corresponding to the symbol $X(e_i, b, e_j)$. The usual map $\mathrm{St}(B) \to \mathrm{Gl}(B)$ sends $X(v, x, w)$ to $1 + vxw$.

Similarly, $\mathrm{St}(B^\infty, I, B^\infty)$ is isomorphic to $\mathrm{St}(B, I)$, with the symbol $X(e_i, x, e_j)$ corresponding to $y_{ij}(x)$. The symbols $y_{ij}(x)$ generate $\mathrm{St}(B, I)$ as an $\mathrm{St}(B)$-group. (See [5], [6] and [7].) In fact, there is a $\mathrm{Gl}(B)$-group structure on $\mathrm{St}(B^\infty, I, B^\infty)$ given by

$$g \cdot X(v, x, w) = X(gv, x, wg^{-1}) \qquad (g \in \mathrm{Gl}(B)).$$

We can create variant Steinberg groups using the 2-sided submodule $I^\infty = IB^\infty = B^\infty I$ of B^∞. The group $\mathrm{St}(I^\infty, B, I^\infty)$ of Theorem 4 is such a variant. Note that the image of the map

$$\mathrm{St}(I^\infty, B, I^\infty) \to \mathrm{St}(B^\infty, B, I^\infty) \to \mathrm{Gl}(B, I)$$

is the subgroup $E(I^\infty, 1, I^\infty)$ of $\mathrm{Gl}(B, I)$. The following result connects some more of these variants:

Theorem 8.3 (van der Kallen). *There are isomorphisms of* $\mathrm{Gl}(B)$*-groups:*

(a) $\mathrm{St}(B, I) \to \mathrm{St}(B^\infty, I, B^\infty)$ *via* $y_{ij}(x) \to X(e_i, x, e_j)$

(b) $\mathrm{St}(B^\infty, I, B^\infty) \to \mathrm{St}(B^\infty, B, I^\infty)$ *via* $X(v, x, w) \to X(v, 1, xw)$

(c) $\mathrm{St}(B^\infty, I, B^\infty) \to \mathrm{St}(I^\infty, B, B^\infty)$ *via* $X(v, x, w) \to X(vx, 1, w)$.

Proof. This is (A8) of [9].

9. Here is an analysis of the groups $\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)$ and $\mathrm{St}(I^\infty, B, I^\infty)$. The map $\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty) \to \mathrm{St}(I^\infty, B, I^\infty)$ is onto, because in $\mathrm{St}(I^\infty, B, I^\infty)$ we have $X(v, x, w) = X(v, 1, xw)$. Therefore the subgroups $\overline{\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)}$ and $\overline{\mathrm{St}(I^\infty, B, I^\infty)}$ of $\mathrm{St}(B, I)$ are equal. Since the map $\mathrm{St}(I^\infty, B, I^\infty) \to \mathrm{St}(B^\infty, B, I^\infty) = \mathrm{St}(B, I)$ is a map of $\mathrm{Gl}(B)$-groups (see [4, (3.3)]), relation 8(f) implies that the image of $\mathrm{St}(I^\infty, B, I^\infty)$ is normal in $\mathrm{St}(B, I)$.

Now suppose that $I^2 = 0$. In this case, the presentation (8) immediately gives $\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty) = I^\infty \otimes_{\mathbb{Z}} I^\infty$ and $\mathrm{St}(I^\infty, B, I^\infty) = I^\infty \otimes_B I^\infty$. Their common image in $\mathrm{St}(B, I)$ can be zero, as the example $B = \mathbb{F}_2[\varepsilon]$, $I = \varepsilon B$ shows. This shows that the abstract groups $\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)$, $\mathrm{St}(I^\infty, B, I^\infty)$ and $\overline{\mathrm{St}(I^\infty, \mathbb{Z}, I^\infty)} = \overline{\mathrm{St}(I^\infty, B, I^\infty)}$ are not isomorphic.

Proposition 10 (Right exactness of St). *The sequence of groups*

$$\mathrm{St}(B, I^2) \to \mathrm{St}(B, I) \to \mathrm{St}(B/I^2, I/I^2) \to 1$$

is exact.

Proof. By [4, (3.5)], the group $\mathrm{St}(B, I)$ is generated by the symbols $y_{ij}(b; x) = X(e_j + e_i b, x, e_i - be_j)$, where $b \in B$, $x \in I$, and $i \neq j$. These map onto the corresponding generators of $\mathrm{St}(B/I^2, I/I^2)$. Hence the map $\psi : \mathrm{St}(B, I) \to \mathrm{St}(B/I^2, I/I^2)$ is onto.

In [5, Theorem 9], Keune proved that St is 'right exact' in the sense that it preserves split coequalizers. We apply this to the split coequalizer

$$(B \oplus I^2, I \oplus I^2) \rightrightarrows (B, I) \to (B/I^2, I/I^2)$$

obtaining the split coequalizer diagram

$$\mathrm{St}(B \oplus I^2, I \oplus I^2) \rightrightarrows \mathrm{St}(B, I) \xrightarrow{\psi} \mathrm{St}(B/I^2, I/I^2).$$

By [4, (3.5)], it follows that $\ker(\psi)$ is generated by the elements

$$y_\alpha(b + z_0; x + z_1) y_\alpha(b; x)^{-1} \qquad (b \in B, x \in I, z_i \in I^2).$$

$$= [y_\alpha(z_0), y_\alpha(b; x + z_1)] y_\alpha(b; z_1)$$

$$=y_\alpha(z_0)\{y_\alpha(b; x+z_1)\cdot y_\alpha(-z_0)\}y_\alpha(b; z_1).$$

These elements lie in the normal subgroup $\overline{\mathrm{St}(B,I^2)}$ because all of their terms do. Hence $\overline{\mathrm{St}(B,I^2)}=\ker(\psi)$, completing the proof.

11. In this paragraph we will establish all of the inclusion relationships in Theorems 3 and 4, and prove 4(iii). The equality of $\overline{\mathrm{St}(B,I^2)}$ with $\ker(\mathrm{St}(B,I)\to\overline{\mathrm{St}(B/I^2,I/I^2)})$ follows from Proposition 10. The inclusion of $\overline{\mathrm{St}(B,I^2)}$ in $\overline{\mathrm{St}(I^\infty,\mathbb{Z},I^\infty)}=\overline{\mathrm{St}(I^\infty,B,I^\infty)}$ follows from the fact that (for $x_i, y_i\in I$) the element $X(v,\sum x_iy_i,w)$ of $\mathrm{St}(B^\infty,I^2,B^\infty)$ and the element $\prod X(vx_i,1,y_iw)$ of $\mathrm{St}(I^\infty,B,I^\infty)$ have the same image in $\mathrm{St}(B^\infty,I,B^\infty)\simeq\mathrm{St}(B^\infty,B,I^\infty)$.

Lemma 11.1. $\overline{\mathrm{St}(I^\infty,B,I^\infty)}\subseteq[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]$.

Proof. Choose a generator $X(v,1,w)$ of $\mathrm{St}(I^\infty,B,I^\infty)$. Pick n so that

$$v=\sum_{i=1}^n e_iv_i,\quad w=\sum_{i=1}^n w_ie_i\qquad(v_i,w_i\in I).$$

Choose $s>n$. Then in $\mathrm{St}(B^\infty,B,I^\infty)$ we have

$$\begin{aligned}\left[\prod_{i=1}^n X(e_i,1,v_ie_s),X(e_s,1,w)\right]&=\{\left(\begin{smallmatrix}1&v\\0&1\end{smallmatrix}\right)\cdot X(e_s,1,w)\}X(e_s,1,w)^{-1}\\&=X[\left(\begin{smallmatrix}1&v\\0&1\end{smallmatrix}\right)e_s,1,w\left(\begin{smallmatrix}1&-v\\0&1\end{smallmatrix}\right)]X(e_s,1,w)^{-1}\\&=X(e_s+v,1,w)X(e_s,1,w)^{-1}\\&=X(v,1,w).\end{aligned}$$

This shows that $X(v,1,w)\in[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]$ and hence that $\overline{\mathrm{St}(I^\infty,B,I^\infty)}\subseteq[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]$.

Lemma 11.2. $[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]=[\mathrm{St}(B,I),\mathrm{St}(B,I)]=[\mathrm{Gl}(B,I),\mathrm{St}(B,I)]$.

Proof. Using $\overline{\mathrm{St}(I)}\subseteq\mathrm{St}(B,I)$ and relation 8.1(f), we have

$$[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]\subseteq[\mathrm{St}(B,I),\mathrm{St}(B,I)]\subseteq[\mathrm{Gl}(B,I),\mathrm{St}(B,I)].$$

The group $\mathrm{St}(B,I)$ is the direct colimit of the groups $\mathrm{St}(B^n,B,I^n)$. From the identity $[f,gh]=[f,g]g[f,h]g^{-1}$ it follows that $[\mathrm{Gl}(B,I),\mathrm{St}(B,I)]$ is generated by the elements $[g,X]=(g\cdot X)X^{-1}$ for g in $\mathrm{Gl}_n(B,I)$ and $X=X(v,x,w)$ in $\mathrm{St}(B^n,B,I^n)$. We know that the matrix $\left(\begin{smallmatrix}g&0\\0&g^{-1}\end{smallmatrix}\right)$ lies in $E_{2n}(B,I)$ by [1, p. 227]; lift it to an element Y in $\mathrm{St}(B^{2n},B,I^{2n})$. Then in $\mathrm{St}(B^{2n},B,I^{2n})$ we have

$$[g,X]=[g,X(v,x,w)]=[\left(\begin{smallmatrix}g&0\\0&g^{-1}\end{smallmatrix}\right),X(\left(\begin{smallmatrix}v\\0\end{smallmatrix}\right),x,(w\ 0))]=[Y,X].$$

Next note that the quotient group $\mathrm{St}(B,I)/\overline{\mathrm{St}(I)}$ is abelian by [4, (3.10)], while the quotient group $\mathrm{St}(B,I)/\mathrm{St}(B,I)\cap\mathrm{Gl}(I^2)$ is abelian because it is isomorphic to its image in the abelian group $\mathrm{Gl}(I)/\mathrm{Gl}(I^2)\subseteq\mathrm{Gl}(I/I^2)$. This shows that

$$[\mathrm{St}(B,I),\mathrm{St}(B,I)]\subseteq\overline{\mathrm{St}(I)}\cap(\mathrm{St}(B,I)\cap\mathrm{Gl}(I^2))=\overline{\mathrm{St}(I)}\cap\mathrm{Gl}(I^2).$$

Finally, it is obvious that

$$\overline{K_2(\mathbb{Z}\oplus I,I)}\subseteq\overline{\mathrm{St}(I)}\cap\mathrm{Gl}(I^2)\subseteq\mathrm{St}(B,I)\cap\mathrm{Gl}(I^2).$$

11.3. If we map $\mathrm{St}(B,I)$ onto $E(B,I)$, we immediately obtain all the inclusion relationships claimed in Theorem 2 except for the fact that $[\mathrm{Gl}(B,I),\mathrm{Gl}(B,I)]$ is contained in $\mathrm{Gl}(I^2)$. This is immediate from the fact that $\mathrm{Gl}(I)/\mathrm{Gl}(I^2)$ is abelian.

Proposition 12. *When* $I^2=0$,

$$\overline{\mathrm{St}(I^\infty,B,I^\infty)}=[\overline{\mathrm{St}(I)},\overline{\mathrm{St}(I)}]=[\mathrm{St}(B,I),\mathrm{St}(B,I)]$$
$$=[\mathrm{Gl}(B,I),\mathrm{St}(B,I)].$$

Proof. The inclusion relations were established in 11. For $g\in\mathrm{Gl}(B,I)$ and $X(v,x,w)\in\mathrm{St}(B^\infty,B,I^\infty)=\mathrm{St}(B,I)$ we have

$$g\cdot X(v,x,w)=X(gv,x,wg^{-1})=X(gv,x,w),$$
$$[g,X(v,x,w)]=X(gv,x,w)X(v,x,w)^{-1}=X((gv-v),x,w).$$

This element is in the normal subgroup $\overline{\mathrm{St}(I^\infty,B,I^\infty)}$ of $\mathrm{St}(B,I)$. From the identity $[f,gh]=[f,g]g[f,h]g^{-1}$ it follows that every generator $[g,h]$ of $[\mathrm{Gl}(B,I),\mathrm{St}(B,I)]$ belongs to $\overline{\mathrm{St}(I^\infty,B,I^\infty)}$.

Example 13. Even when $I^2=0$ the group $[\mathrm{St}(B,I),\mathrm{St}(B,I)]$ can be nontrivial. To see this, note that the following relation holds in $\mathrm{St}(B^\infty,B,I^\infty)\cong\mathrm{St}(B,I)$:

$$X(e_1x,1,ye_1)=X(e_1x+e_2,1,ye_2)X(e_2,1,ye_1)^{-1}$$
$$=\{\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\cdot X(e_2,1,ye_1)\}X(e_2,1,ye_1)^{-1}$$
$$=[y_{12}(x),y_{21}(y)]$$
$$=\langle x,y\rangle\in K_2(B,I)\qquad (x,y\in I,\ I^2=0).$$

This element can be nonzero [7], [2]. Indeed, if $B=R\oplus I$ is commutative and $I^2=0$, then by [10, (1.4)] there is a surjection from $K_2(B,I)$ to $\Lambda^2 I$ sending $\langle x,y\rangle$ to $x\wedge y$.

Proof of Theorem 3. When $I^2=0$, $\mathrm{St}(B,I)$ is an extension of the abelian group $E(B,I)$ by the abelian group $K_2(B,I)$, and hence is nilpotent. Example 13 shows that $\mathrm{St}(B,I)$ need not be abelian. The second sentence of Theorem 3 follows from 11, 12 and from the observation in [2, (6.4)] that when $I^2=0$ the group $K_2(\mathbb{Z}\oplus I,I)=\mathrm{St}(I)\cap\mathrm{Gl}(I^2)$ is generated by elements in $[\mathrm{St}(I),\mathrm{St}(I)]$. Finally, from [4, (0.2)] we see that the cokernel of $K_2(\mathbb{Z}\oplus I,I)\to K_2(B,I)$ is the Hochschild homology group $H_1(B;I)$, which is generally nonzero.

Proposition 14. *For every ideal I*

$$[\overline{\mathrm{St}(I)}, \overline{\mathrm{St}(I)}] = [\mathrm{St}(B, I), \mathrm{St}(B, I)] = [\mathrm{Gl}(B, I), \mathrm{St}(B, I)]$$
$$= \overline{\mathrm{St}(I)} \cap \mathrm{Gl}(I^2).$$

In particular, $\overline{K_2(\mathbb{Z}\oplus I, I)}$ *lies in the commutator subgroup of* $\mathrm{St}(B, I)$.

Proof. Let $A = \mathbb{Z}\oplus I$. Using the sequence in Proposition 10 and the equalities of Theorem 3, we obtain the diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & \downarrow & & & & \\
\mathrm{St}(A, I^2) & \longrightarrow & [\mathrm{St}(A, I), \mathrm{St}(A, I)] & \longrightarrow & [\mathrm{St}(A/I^2, I/I^2), \mathrm{St}(A/I^2, I/I^2)] & \longrightarrow & 1 \\
\| & & \downarrow & & \| & & \\
\mathrm{St}(A, I^2) & \longrightarrow & \mathrm{St}(A, I) \cap \mathrm{Gl}(I^2) & \longrightarrow & K_2(A/I^2, I/I^2) & \longrightarrow & 1
\end{array}$$

A diagram chase shows that $[\mathrm{St}(I), \mathrm{St}(I)] = \mathrm{St}(I) \cap \mathrm{Gl}(I^2)$ as subgroups of $\mathrm{St}(I) = \mathrm{St}(A, I)$. Hence in $\mathrm{St}(B, I)$ we have

$$[\overline{\mathrm{St}(I)}, \overline{\mathrm{St}(I)}] = \overline{\mathrm{St}(I) \cap \mathrm{Gl}(I^2)} = \overline{\mathrm{St}(I)} \cap \mathrm{Gl}(I^2).$$

Proof of Theorem 4. Part (i) follows from Proposition 10, and part (ii) was proven in 9. Parts (iii) and (iv) were shown in Proposition 14. The chain of inclusions was established in 11. Normality of $\mathrm{St}(I^\infty, B, I^\infty)$ was demonstrated in 9. Since $\mathrm{St}(B, I^2) \to \mathrm{St}(B, I)$ is a map of $\mathrm{Gl}(B)$-groups, the image is normal in $\mathrm{St}(B, I)$. Normality of the image of $\mathrm{St}(I) \to \mathrm{St}(B, I)$ was established in [4,(3.8)]. To see that the inclusions can be strict, map $\mathrm{St}(B, I)$ onto $E(B, I)$ and refer to Theorem 2.

References

[1] H. Bass, Algebraic *K*-theory (Benjamin, New York, 1968).

[2] R.K. Dennis and M. Krusemeyer, $K_2(A[X, Y]/XY)$, a problem of Swan and related computations, J. Pure Appl. Algebra 15 (1979) 125–148.

[3] S. Geller and L. Roberts, K_2 of some truncated polynomial rings, Lecture Notes in Math. 734 (Springer, Berlin, 1979).

[4] S. Geller and C. Weibel, $K_1(A, B, I)$, J. Reine Angew. Math.

[5] F. Keune, The relativization of K_2, J. Algebra 54 (1978) 159–177.

[6] J.-L. Loday, Cohomologie et groupe de Steinberg relatifs, J. Algebra 54 (1978) 178–202.

[7] R. Swan, Excision in algebraic *K*-theory, J. Pure Appl. Algebra 1 (1977) 221–252.

[8] L. Vaserstein, On the stabilization of the general linear group, Math. USSR Sbornik 8 (1969) 383–400. (= Mat. Sbornik 79 (1969) 405–424.

[9] W. van der Kallen, Appendix to "Localization of the *K*-theory of polynomial extensions" (by T. Vorst), Mat. Annalen 244 (1979) 51–53.

[10] C. Weibel, K_2, K_3 and nilpotent ideals, J. Pure Appl. Algebra 18 (1980) 333–345.